Garegnani, P. 1984. Value and distribution in the classical economists and Marx. *Oxford Economic Papers* 36(2), 291–325.

Manara, C.F. 1979. Sraffa's model for the joint production of commodities by means of commodities. In *Essays on the Theory of Joint Production*, ed. L.L. Pasinetti, New York: Columbia University Press.

Marx, K. 1867, 1885, 1894. *Capital*, Vols. I, II, III. English translations in C.H. Kerr (ed.), Chicago: Chicago University Press, or other editions.

Metcalfe, J.S. and Steedman, I. 1972. Reswitching and primary input use. *Economic Journal* 82, March, 140–57.

Miyao, T. 1977. A generalization of Sraffa's standard commodity and its complete characterization. *International Economic Review* 18, February, 151–62.

Neumann, J. von. 1945. A model of general economic equilibrium. In *Review of Economic Studies* 13, 1–9. Trans. of an article in *Ergebnisse eines mathematischen Kolloquiums*, Vol. 8, ed. K. Menger, Leipzig, 1937.

Pasinetti, L.L. 1977. *Lectures on the Theory of Production*. New York: Columbia University Press.

Robinson, J. 1956. *The Accumulation of Capital*. London: Macmillan.

Samuelson, P.A. 1975. Marx as mathematical economist: steady-state and exponential growth equilibrium. In *Trade, Stability, and Macroeconomic, Essays in Honor of Lloyd A. Metzler*, ed. G. Horwich and P. Samuelson, New York: Academic Press, 269–307. Also ch. 225 in *The Collected Scientific Papers of Paul A. Samuelson*, Vol. III, Cambridge, Mass.: MIT Press, 1972.

Samuelson, P.A. 1975. Trade pattern reversals in time-phased Ricardian systems and intertemporal efficiency. *Journal of International Economics* 5, November, 309–63. Also ch. 251 in *The Collected Scientific Papers of Paul A. Samuelson*, Vol. IV, Cambridge, Mass.: MIT Press, 1977.

Samuelson, P.A. 1983. Thünen at two hundred. *Journal of Economic Literature* 21, December, 1468–88.

Schefold, B. 1971. *Theorie der Kuppelproduktion (Mr. Sraffa on Joint Production)*. Privately printed, Basel.

Seton, F. 1957. The transformation problem. *Review of Economic Studies* 25, 149–60.

Sraffa, P. 1926. The laws of returns under competitive conditions. *Economic Journal* 36, December, 535–50.

Sraffa, P. (ed.) 1951–73. *The Works and Correspondence of David Ricardo*. Cambridge: Cambridge University Press, for the Royal Economic Society; with editorial collaboration by M.H. Dobb.

Sraffa, P. 1960. *Production of Commodities by Means of Commodities: Prelude to a Critique of Economic Theory.* Cambridge: Cambridge University Press.

Steedman, I. 1977. *Marx After Sraffa*. London: New Left Books. Reprinted, London: Verso, 1981.

Steedman, I. 1982. Marx on Ricardo. In *Classical and Marxian Political Economy: Essays in Honour of Ronald L. Meek*, ed. I. Bradley and M. Howard, London: Macmillan.

Stigler, G.J. 1958. Ricardo and the 93% labor theory of value. *American Economic Review* 48, June, 357–67.

Sraffian economics (new developments)

1 Introduction

In an earlier era, Sraffians took aim at the neoclassical assertion that the demand for and supply of labour and other resources determine factor incomes. In stalking the big game, a smaller question served as bait: is there is a homogenous substance, aggregate capital, whose marginal product determines the return to capital? After a brief episode of disagreement in the 1960s, the neoclassical side conceded that in models with even a minimal disaggregation of capital goods the answer to the smaller question was 'no'. Despite this bloodletting, the chase petered out. When the hunter and hunted ran out of formal modeling disagreements, a settlement was drawn up.

The agreement stipulated that there are two neoclassical theories. The first is an aggregative model that tells the familiar parables of Solow growth theory: increases in savings raise the ratio of the value of capital to labour employed, the rate of interest falls as production becomes more capital intensive, and so on. But once a multiplicity of capital goods is introduced these parables no longer hold true. General equilibrium theory on the other hand places no limit on the number of capital or consumption goods and still gives a coherent, determinate account of markets and price determination and hence of the distribution of factor incomes.

While both parties could agree to this decree, they took different views as to who walked away with the more valuable share of the community property. By the time of the split in the early 1970s, general equilibrium had already been singled out as the jewel of microeconomic theory. The results that had to be jettisoned were confined to the steady-state effects of capital accumulation, leaving the prize results of general equilibrium theory – the existence of equilibrium and its welfare properties – untouched. Moreover, a multiplicity of consumption goods will by itself (without multiple capital goods) imply that the distribution of income cannot be determined by *the* marginal products of capital and labour. The disavowal by high theory of aggregate capital therefore seemed a small loss. Hahn (1982), which marked the end of engagement with Sraffian dissent, argued not only that general equilibrium theory left the line of neoclassical succession intact but also that all legitimate Sraffian results could be obtained from suitably specialized general equilibrium models. The resilience of general equilibrium theory to the Sraffian assault had the curious effect that aggregative empirical work could proceed unaffected by the Sraffa episode. Even a literature such as growth accounting for which the Sraffa critique was pertinent showed no influence – it took off just as the Sraffa attack reached its height. Since general equilibrium theory viewed all forms of aggregation as suspect, overlooking Sraffian concerns about capital aggregation seemed to be one of the ordinary compromises that applied work demands.

The Sraffian reaction was more complicated. Some held that modern general equilibrium theory, although internally consistent, fails because it does not explain how relative prices converge through time to long-run values, in this view the proper goal of economic science (Garegnani, 1976; Eatwell, 1982; Kurz and Salvadori, 1995). Another strand simply promoted Sraffian and classical economics more generally as a distinct type of economic theory (for example, Harcourt, 1974; Marglin, 1984). Sraffians take as their starting place a wage or wage-share of output that is determined by non-economic forces, for example by political power or by social consensus. The general equilibrium model in contrast explains factor prices as the outcome of the endogenous play of supply and demand. Different assumptions, different theories: let the evidence decide.

An agreement to disagree should leave all parties dissatisfied. If Sraffian economics and general equilibrium theory are merely two contenders, each with its own starting place about the causal forces that move factor prices, to be adjudicated by empirical test, then all the wrangling in the 1960s was for nought. For if wages are determined, by political power, say, rather than supply and demand, then political power could remain the prime determinant even if capital always aggregated perfectly or in models where prices do not converge to long-run values. What makes the Sraffa–neoclassical debate significant is its critical dimension, the Sraffian arguments that the forces of competition cannot pin down the distribution of income, that any supply and demand theory is riddled with internal flaws. The neoclassical side of the debate has contributed its share of confusion: the mere existence of competitive equilibria does not speak to the adequacy of a supply-and-demand account of factor incomes. When translated into the language of general equilibrium, the Sraffian complaints presumably concern the determinacy and stability of equilibrium, not existence or optimality. Yet Hahn (1982), for example, treats determinacy only casually and leaves stability unaddressed. Fortunately, some decades of delay after the noisy 1960s and 1970s, the literature on Sraffa has turned to these points.

The Sraffian complaints about supply-and-demand theories of price determination can be spelled out in two ways. The first appears in Sraffa's *Production of Commodities by Means of Commodities* (1960): the laws of competitive markets do not fully determine factor prices or the distribution of income. Competition requires that the same rate of return is earned in every sector; when laid out as a system of equations, this requirement leaves one more variable than equation, thus revealing a single dimension of indeterminacy, or, as Hahn (1982) put it, a 'missing equation'. Hahn and other neoclassical economists responded that the missing equation would be found as soon as supply-equals-demand equalities are incorporated into Sraffa's model. Sraffians vacillate on market clearing for factors of production: the land market has to clear, but the labour market does not. This asymmetry drives the single dimension of indeterminacy. As we will see, if the same market-clearing conditions that Sraffians impose on land markets to quash extra dimensions of indeterminacy are applied to the labour market then even the standard single dimension of indeterminacy can disappear. This conclusion would seem to undercut Sraffa's book: if indeterminacy stems solely from failing to require the labour market to clear then Sraffians hardly need an elaborate model to press their point. In essentially any setting, the deletion of labour-market clearing will turn the wage into a free variable and hence leave the distribution of income indeterminate. On this score at least, there would be no need to object to the aggregate neoclassical production function.

But the story is not so simple: the neoclassical presumption that the full gamut of market-clearing conditions necessarily brings determinacy is not correct. Although the ingredients have to be recombined, the Sraffian tradition takes just the right modeling steps that lead factor prices in general equilibrium models to be indeterminate. Sraffians have long insisted that linear activities provide a more faithful representation of technology than the differentiable production functions that dominate practical work in neoclassical economics. Although linear activities by themselves do not generate indeterminacy, they can when endowments of capital goods are governed by rational savings decisions rather than by chance. The Sraffian view of the economy as an ongoing cycle of reproduction thus paves the way for factor-price indeterminacy. On the other hand, the particular way Sraffa and his followers have spelled out their long-run view of the economy, by requiring that relative prices be constant through time, undermines their 'missing equation' criticism: linear activities models with constant relative prices have determinate factor prices. And the aggregation of capital has no bearing on the matter: the determinacy of a supply-and-demand theory of factor incomes depends on how many activities operate compared with the number of scarce factors, not on the number of capital goods.

The second completion of the Sraffa critique focuses on the potential for the value of capital per worker to behave badly, for example to increase in response to a rise in the interest rate. Although it might seem that this possibility could by itself lead to instability, this turns out not to be the case. Instability can arise in general equilibrium but it stems from the demand side of the model, not the failure of capital goods to aggregate.

Little of the Sraffian–neoclassical settlement therefore withstands scrutiny. While a couple of assertions in Solow growth theory about steady states hinge on whether the economy has a single sector and whether capital aggregates, the operation of competitive markets does not. The neoclassical confidence that the general equilibrium model answers all Sraffian challenges is equally misplaced: the Sraffian indeterminacy thesis can be reclaimed. As for neoclassical growth theory, its main message that the return to saving diminishes as savings increase can be re-expressed to avoid the limitations of single-sector models. But here too the Sraffian tradition points the way to important corrections. The characteristic neoclassical equality between an economy's interest rate and its marginal rate of transformation is an artifact of differentiable technologies. With linear activities this equality need not obtain, although for the failure of the neoclassical maxim to be robust, we must follow the neoclassical program, rejected by some Sraffians, of letting utility functions determine consumption.

This article's focus on determinacy and stability aims to complement Paul Samuelson's SRAFFIAN ECONOMICS, which draws the lessons to be learned from Sraffa regarding aggregative parables.

2 The single dimension of Sraffian indeterminacy

We set a benchmark model of linear activities. Let there be N goods, one type of labour, L types of land, and finitely many activities. Each activity i when operated at the unit level requires an investment one period in advance of $a_i = (a_{1i}, \ldots, a_{Ni}) \geq 0$ units of the N material inputs and then a contemporaneous application of $\ell_i \geq 0$ units of labour and $\Lambda_i = (\Lambda_{1i}, \ldots, \Lambda_{Li}) \geq 0$ units of the L land types to produce the outputs $b_i = (b_{1i}, \ldots, b_{Ni}) \geq 0$. The level at which activity i is operated is given by y_i. We assume to begin that the prices $p = (p_1, \ldots, p_N) \geq 0$ of the material goods purchased as inputs equal the prices of the same material goods when sold as outputs one period later. Profit maximization requires that no activity makes positive economic profits and that any activity i in use ($y_i > 0$) makes zero economic profits. Let r be the intertemporal interest rate, $w \geq 0$ be the wage, and $\rho = (\rho_1, \ldots, \rho_L) \geq 0$ be the rental rates on land. So profit maximization dictates, for each activity i,

$$\begin{aligned} p_1 b_{1i} + \ldots + p_N b_{Ni} \leq (1+r)(p_1 a_{1i} + \ldots \\ + p_N a_{Ni}) + w\ell_i + \rho_1 \Lambda_{1i} + \ldots \\ + \rho_L \Lambda_{Li} \end{aligned} \tag{2.1}$$

and that equality holds for any i such that $y_i > 0$.

The Sraffa literature equivocates on market clearing for resources. While land types are required to have a 0 rental rate when in excess supply, the situation for labour is often left unspecified. Let e_{Λ_k} denote the supply of type k land and e_ℓ denote the supply of labour. We then have the market-clearing conditions

$$\begin{aligned} &\sum_i \Lambda_{ki} y_i \leq e_{\Lambda_k} \\ &\sum_i \Lambda_{ki} y_i < e_{\Lambda_k} \Rightarrow \rho_k = 0, \end{aligned} \tag{2.2}$$

for land type $k = 1, \ldots, L$. The analogous conditions for labour are

$$\sum_i \ell_i y_i \leq e_\ell, \qquad \sum_i \ell_i y_i < e_\ell \Rightarrow w = 0.$$

Letting y denote a vector activity levels, an *equilibrium* is a $(p, w, 1+r, y)$ that satisfies (2.1) and (2.2). When we impose labour-market clearing, we say so explicitly. In the background lurk additional market-clearing conditions for produced goods, which we introduce in Section 3.

Since we are interested only in relative prices, we can normalize prices by choosing one of the goods or a bundle of goods as numéraire. We set

$$p_1 + \ldots + p_N = 1. \tag{2.3}$$

In the basic Sraffa model, the focus of the first generation of literature, each activity produces only one good and uses no land, and only one activity is available to produce each good j and is therefore given the index j. So in this case we can rescale each (b_i, a_i, ℓ_i) so that b_{ii} (the sole non-zero coordinate of b_i) equals 1. Assuming that each activity is in use, then (2.1) gives us the classical Sraffa equalities:

$$\begin{aligned} p_i = (1+r)(p_1 a_{1i} + \ldots + p_N a_{Ni}) + w\ell_i, \\ i = 1, \ldots, N. \end{aligned} \tag{2.4}$$

The simplest version of Sraffa's missing equation or single dimension of indeterminacy thesis amounts to the observation that (2.4) and the normalization (2.3) comprise $N+1$ equations but contain $N+2$ price variables $(p, w, 1+r)$. A complete argument that any $(p, w, 1+r) \gg 0$ that satisfies (2.3) and (2.4) is locally contained in a one-dimensional set of points that solve the same equalities might seem to require a rank condition, but the linearity and homogeneity of the model make this unnecessary (see the *Note on the dimension of indeterminacy of the basic Sraffa model* at the end of the text). Since typically we can parameterize the solutions to (2.3) and (2.4) by w or r, the distribution of income is indeed indeterminate. Competition, Sraffians suggest, does not pin down a division of social wealth between capital and labour.

The indeterminacy of the above model has little significance from the neoclassical point of view: the only alarming possibility would be if market-clearing equalities for some reason could not close the model. The Sraffian literature does not engage this argument, however, and instead takes either w or r to be exogenous, set by political factors or by a macroeconomic determination of the interest rate. The rationale for this practice is perpetually unclear: is it that market-clearing equalities cannot fill the indeterminacy gap or does some principle trump the laws of supply and demand?

We now document the efforts of Sraffa and his followers to maintain precisely a single dimension of indeterminacy. 'Single dimension of indeterminacy' is not standard terminology; a more conventional but equivalent description would be that Sraffians aim to show that (2.1) and (2.3) locally determine prices once w or r has been set, and hence that, given w or r and given an equilibrium, a small exogenous change in demand will leave prices unaffected. Indeed, the view that prices in the long-run are affected only by technology and the distribution of income, not the composition of demand, has long been a top item of the Sraffian theoretical agenda.

Single-dimensional indeterminacy faces three threats – rent, joint production, and the choice of technique – the same topics that dominate the second half of Sraffa's book and the second generation of the Sraffian literature. Although we will see that the arguments available against extra dimensions of indeterminacy can sometimes be turned against the presence of any indeterminacy at all, our position is that zero, one, and more than one are all plausible equilibrium possibilities for the dimension of indeterminacy. It may seem in some of our exhibits that demand and market-clearing should eradicate any indeterminacy, but Section 3 will show that they are compatible.

Exhibit A: land and rent

If we add an additional non-produced factor with a positive price to a Sraffa model the number of endogenous price variables will increase. If no further activities are drawn into production, the dimension of indeterminacy will normally go up.

For an elementary example, let there be one produced good ($N = 1$), one type of land ($L = 1$) with endowment $e_\Lambda > 0$, and multiple activities. Let the rental rate of the single land type be ρ and again normalize activities so that when activity i is operated at the unit level one unit of output is produced. Suppose activity i with $(a_{1i}, \ell_i, \Lambda_i) \gg 0$ is in use and $y_m = 0$ for $m \neq i$. Then (2.1) reduces to

$$1 = (1 + r)a_{1i} + w\ell_i + \rho\Lambda_i, \qquad (2.5)$$

$$1 \leq (1 + r)a_{1m} + w\ell_m + \rho\Lambda_m, \quad m \neq i, \qquad (2.6)$$

where we have substituted in our normalization $p_1 = 1$. To ensure that (2.2) does not constrain ρ to equal 0, the land must be fully employed: $\Lambda_i y_i = e_\Lambda$. With ρ as an additional free variable, if $(\bar{w}, 1 + \bar{r}, \bar{\rho}) \gg 0$ satisfies (2.5) and (2.6) and each inequality in (2.6) is strict, an additional dimension of indeterminacy obtains: if we independently vary w and $1 + r$ a small amount then ρ can be adjusted so that (2.5) and (2.6) remain satisfied.

Another type of equilibrium occurs when two or more activities are in use. If two activities i and j are in use, then (2.5) is replaced by two equalities,

$$\begin{aligned} 1 &= (1 + r)a_{1i} + w\ell_i + \rho\Lambda_i, \\ 1 &= (1 + r)a_{1j} + w\ell_j + \rho\Lambda_j. \end{aligned} \qquad (2.7)$$

Condition (2.6) now holds for $m \neq i, j$, and full employment for land is given by $\Lambda_i y_i + \Lambda_j y_j = e_\Lambda$. Evidently the argument for an additional dimension of indeterminacy now fails; we cannot independently vary w and $1 + r$ and expect to satisfy both equalities in (2.7) using the single free variable ρ.

The Sraffa literature largely focuses on the second type of equilibrium with a single dimension of indeterminacy. Is this the more likely type? To make the best case, notice that in the first type of equilibrium the produced input must be accumulated in an amount that leads the stock of land to be fully utilized using only activity i – otherwise (2.2) would require $\rho = 0$. If e_c is the stock of the produced input accumulated each period, then to fully employ both e_c and the entire land supply e_Λ using only activity i there must be an activity level $y_i \geq 0$ such that $a_{1i}y_i = e_c$ and $\Lambda_i y_i = e_\Lambda$. Since e_c must therefore equal $e_\Lambda a_{1i}/\Lambda_i$, perhaps one could conclude that the accumulation of this exact amount is unlikely to occur. But consider how the shape of the production possibilities set changes as e_c changes. If we fix w arbitrarily (and suppose implicitly there is no constraint on labour supply), then, outside of exceptional values for w and barring flukes in the input usage coefficients, at most two activities can have the least cost per unit of output and be employed by profit-maximizing firms. If exactly two activities are in use, then the economy can raise its usage of the produced input and increase output by switching the mix of the two activities towards whichever activity economizes on the use of land and uses the produced input intensively – the activity j with the higher a_{1j}/Λ_j ratio. This remixing delivers a linear increase in current output as e_c rises. Since increases in e_c must come from the previous period, the production possibilities frontier (PPF) for the current and previous period's consumption is also linear at points where two activities are in use. Once remixing has been exhausted (a 'switch point'), a new activity with a higher a_{1j}/Λ_j must be adopted if more capital is to be used to produce more current output. At the switch point, the first type of equilibrium occurs where one activity is in use and the PPF exhibits a kink (non-differentiability). But optimizing agents may well choose to save a quantity of the produced input so as to end up at a kink in the PPF. See Figure 1, which pictures the tangency between a kink in a PPF and a smooth indifference curve, where consumption at other periods is fixed at optimal levels. (In a multi-agent model, one may interpret the indifference curve as the boundary of the set of consumptions that can Pareto improve on the optimum.) Such one-activity-in-use

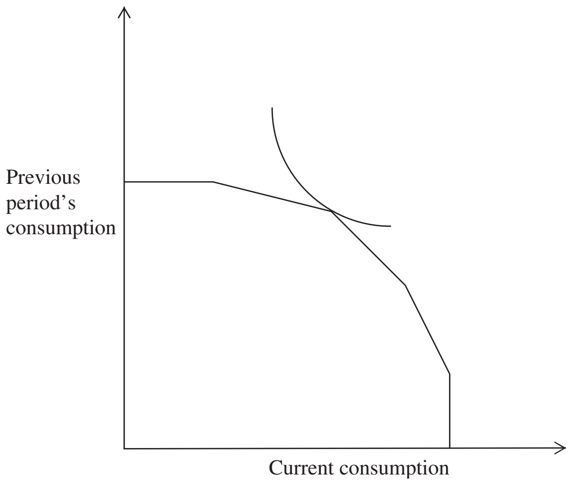

Figure 1 Production possibility frontier and indifference curve

optima are robust to perturbations of the model. Hence, contrary to Sraffian practice, the first type of equilibrium with the extra dimension of indeterminacy should not be excluded.

We will see in Exhibit C that if nevertheless we dismissed the first type of equilibrium as unlikely then we would be compelled also to dismiss equilibria with the traditional Sraffian single dimension of indeterminacy. The view that a single dimension of indeterminacy obtains across modeling environments therefore cannot be upheld.

The Sraffian theory of rent often considers cases where a given type of land is used in only one sector out of many. A different rationale is then available for concluding that a single dimension of indeterminacy will be the norm. Let $N > 1$ and suppose some good j is produced by single-output activities that use various types of land in addition to labour and material inputs. We consider the simplest scenario, extensive rent, where each type of land is used by only one activity (Sraffa, 1960, ch. 11; Quadrio-Curzio, 1980; for more general models, see Salvadori, 1986, and Bidard, 2004, ch. 17). We assign each land type the same index as the activity that uses it. Renormalizing again so that activities produce one unit of output when run at the unit level, the profit-maximization conditions for the production of j appear as

$$p_j \leq (1+r)(p_1 a_{1i} + \ldots + p_N a_{Ni}) + w\ell_i + \rho_i \Lambda_{ii}, \tag{2.8}$$

for each activity i that produces j and where equality holds if $y_i > 0$. The market-clearing requirements (2.2) for land types are assumed to hold. Let the remainder of the economy's sectors satisfy the assumptions of the basic Sraffa model: for each good $k \neq j$, one single-output activity that uses no land is available and in operation. To avoid the distraction of feedback from changes in p_j into j's cost of production, we assume that j does not directly or indirectly enter into the production of any other good. We pick some good besides j as numéraire and fix w. Finally, letting c_i denote the non-land cost of production $(1+r)(p_1 a_{1i} + \ldots + p_N a_{Ni}) + w\ell_i$ for an activity i that produces j, we assume that w and technology coefficients are such that no ties occur among the c_i: if i and m are distinct activities that produce j then $c_i \neq c_m$.

As in the previous $N = 1$ example, the presence of an extra dimension of indeterminacy will depend on the extent of production. But since $N > 1$ we may view the extent of the production of j as a consequence of the demand for j rather than of different levels of accumulation. As increases in demand progressively raise p_j the economy will first use the type i land for which c_i is lowest. When this land type is exhausted p_j must rise further, until the type i for which c_i is second lowest earns the rate of return r, at which point production can expand further. And so on. The supply 'function' thus consists of steps where the 'horizontals' indicate that some type i is partly but not fully utilized and the 'verticals' that a set of land types is fully utilized but that p_j is not yet high enough for the lowest cost of the remaining types to be drawn into production.

On the horizontals the standard one dimension of indeterminacy obtains. If l types of land are used to produce j, the single zero-profit equality in (2.4) for p_j is replaced by l equalities from (2.8). But the additional $l-1$ equalities are matched by $l-1$ additional endogenous rental rates – the one land type that is partly utilized is constrained to have a 0 rental rate. Hence, since (2.4) and (2.3) generate a single dimension of indeterminacy, so do the horizontal equilibria. On the verticals an additional dimension of indeterminacy appears: with l types of land in use, $l-1$ additional zero-profit equalities are again present, but now, since the last type of land to be brought into production is no longer constrained to have a 0 rental rate, there are l additional endogenous factor prices.

The Sraffa literature concentrates on the horizontals rather than the verticals, in line with the Sraffian tradition of taking demand to be exogenous. If the demand for j were completely inelastic – unresponsive to price – then it would be an unlikely accident if this inelastic demand happened to coincide with one of the verticals of the supply function. The horizontals have the added advantage for Sraffians that any small shift in demand will leave the equilibrium at the same step and hence have no effect on any price. From the neoclassical point of view, completely inelastic demand seems far-fetched and does not obtain even when agents have Leontiev utilities. An inelastic demand argument for the horizontal equilibria also sometimes has no bite. When $N = 1$ there is no division of demand into separate outputs; only an inelastic accumulation of the produced input can then allow escape from an extra dimension of indeterminacy.

But in the $N > 1$ case and if we grant an elastic demand function for j, the additional indeterminacy of the vertical equilibria is hardly a reason for worry. More land is brought into cultivation because of demand-led increases in p_j; at a vertical equilibrium the demand for j therefore can pin down p_j and determine each rental rate. While the vertical equilibria dash the Sraffian hope to show that demand is locally irrelevant for price determination, they have no broader significance. The ease with which demand disposes of additional dimensions of indeterminacy underscores the pressing need for demand functions in the Sraffa model; without explicit demands, we will never be able to check whether any apparent case of indeterminacy is the genuine article.

Outside of our attention to extra dimensions of indeterminacy, the above account of rent stays close to Sraffa (1960). Sraffa pays heed to the supply-and-demand restrictions on rental rates: he complies with the rule that factors in excess supply must have a zero price and argues that when the scale of production expands the price of a scarce factor used in production should increase. While Sraffa applies these principles only to land, they pertain to labour as well, as we will see in Exhibit C.

Exhibit B: joint production

The simplest case of joint production occurs when $N = 2$ and there is no land. Profit maximization then requires

$$p_1 b_{1i} + p_2 b_{2i} \leq (1 + r)(p_1 a_{1i} + p_2 a_{2i}) + w\ell_i, \qquad (2.9)$$

for each activity i and with equality if $y_i > 0$. We again consider two types of equilibria. In the first type, one activity i is in use and each of the unused activities satisfies (2.9) with strict inequality. Then just one equality constrains the four prices $(p_1, p_2, w, 1 + r)$, and so, given the normalization (2.3), there are two dimensions of indeterminacy. In the second type of equilibrium, two activities are in use, in which case just the standard single dimension of Sraffian indeterminacy obtains.

Dual to the dimensions of indeterminacy in the two types of equilibria are the dimensions of possible net productions of goods. In first type, the net production in a steady state must lie in the one-dimensional cone

$$\{((b_{1i} - a_{1i})y_i, (b_{2i} - a_{2i})y_i) : y_i \geq 0\}$$

while in the second type, with activities i and j active, net production lies in the two-dimensional cone

$$\{((b_{1i} - a_{1i})y_i + (b_{1j} - a_{1j})y_j, (b_{2i} - a_{2i})y_i + (b_{2j} - a_{2j})y_j) : (y_i, y_j) \geq 0\}.$$

The first type of equilibrium might therefore seem implausible: if labour is inelastically supplied, then this supply determines y_i and hence pins down exactly one vector of net outputs in the one-dimensional cone. But this restriction does not undermine the one-activity equilibria; there is ample room for the relative price p_1/p_2 to equilibrate demand to the fixed supply. While the one-activity equilibria therefore cannot be dismissed as pathological, it would seem, as in the extensive rent example in Exhibit A, that the additional indeterminacy will vanish once we introduce explicit market-clearing conditions. See the 'sheep' example in SRAFFIAN ECONOMICS for an illustration of how an economy can move from one type of equilibrium to the other as demand shifts. Many Sraffians contend that the two-activity equilibria – called 'square' since the number of activities in use equals the number of goods – are more likely (Steedman, 1976; Schefold, 1978a; 1978b; 1988; Lippi, 1979). One interesting rationale for this view, Schefold (1990), argues that, if agents always consume goods in fixed proportions, as with Leontiev utilities, then price adjustment will not be able to bring demand in line with supply in the one-dimensional cone. But if the fixed proportions of consumption vary from person to person, then a change in p_1/p_2 will have a differential effect on the scale of consumption of dissimilar agents. The ratio of demands for the two outputs will then vary with p_1/p_2 and equilibration can occur (Salvadori, 1982; 1990; Bidard, 1997; and see also the Samuelson–Schefold interchange in Bharadwaj and Schefold, 1990, and for an overview of the extensive literature Salvadori and Steedman, 1988).

Exhibit C: choice of activities

So far, we have considered how an extra dimension of indeterminacy can arise. When a choice of activities is available to produce one or more goods even the standard single dimension of Sraffian indeterminacy can disappear. Let each activity produce just one good. We suppose there is no land and consider equilibria – $(p, w, 1 + r, y)$ that satisfy (2.1) and (2.3) – where each good is produced. If we ignore whether the labour market clears, the standard single dimension of Sraffian indeterminacy will obtain and we may index equilibrium prices $(p, w, 1 + r)$ by w. For most values of w, and if we bar flukes of the production coefficients, the resulting equilibrium $(p, w, 1 + r)$ will permit exactly N activities to satisfy their 0-economic profit conditions, that is, earn exactly the rate of return r. If an additional $N + 1st$ activity were required to satisfy its 0-profit condition, then the prices $(p, w, 1 + r)$ that satisfy (2.1) and (2.3) will be pinned down. Hence, with a menu of only finitely many activities available, there are only finitely many w at which $N + 1$ zero-profit equalities could be satisfied at an equilibrium obeying (2.1) and (2.3) – these are the 'switch points' at which the economy moves from one set of N cost-minimizing activities to another. Since there are only finitely many switch points, the required values for w might seem like flukes. But once we impose market clearing for labour, equilibrium can demand a switch point and Sraffian indeterminacy then disappears.

For an example, let $N = 1$ and thus by normalization $p_1 = 1$. Suppose two activities are available with input coefficients (a_{11}, ℓ_1) and (a_{12}, ℓ_2). If both activities are in use, then the profit-maximization condition (2.1) and the normalization (2.3) reduce to the two equalities

$$1 = (1 + r)a_{11} + w\ell_1, \qquad 1 = (1 + r)a_{12} + w\ell_2. \tag{2.10}$$

Flukes in coefficients aside, (2.10) will determine a unique $(w, 1 + r)$. In contrast, if only one activity is in use and the idle activity makes strictly negative profits (its inequality in (2.1) is strict), then the standard single dimension of Sraffian indeterminacy obtains. But now consider market clearing, which we impose on both labour and the material input. If labour is inelastically supplied in the quantity e_ℓ and if e_c is the amount of the material input accumulated each period, then full employment of the input and labour requires

$$a_{11}y_1 + a_{12}y_2 = e_c, \qquad \ell_1 y_1 + \ell_2 y_2 = e_\ell, \tag{2.11}$$

where if $y_i > 0$ then activity i satisfies its zero-profit condition with equality. Evidently equilibria with both activities in use can be robust (they are not accidents of the parameters). For typical values of the model's parameters – e_c, e_ℓ, and the a_{ij} coefficients – (2.10) and (2.11) will have a unique solution $(\bar{w}, 1 + \bar{r}, \bar{y})$ and any small variation in the parameters will through a small adjustment of $(w, 1 + r, y)$ lead to a new unique solution. So if we begin with a model that has a two-activity equilibrium $(\bar{w}, 1 + \bar{r}, \bar{y})$ that is strictly positive in each coordinate then as the model is perturbed a two-activity equilibrium will continue to be present. Marginal products for e_c and e_ℓ are also well-defined at these equilibria and equal $(w, 1 + r)$. We have taken the savings/accumulation level e_c to be exogenous, but we could let e_c be a function of the prices $(w, 1 + r)$ without affecting the robustness of the two-activity equilibria. Indeed, a two-activity equilibrium could well be the unique equilibrium – as when, for example, the accumulation level e_c is increasing in r. (That is, if $(1 + r, w)$ and $(1 + r', w')$ both satisfy (2.10) and $r > r'$ then e_c is strictly larger with $(1 + r, w)$ than with $(1 + r', w')$.) The robustness argument in no way hinges on there being a single material good. With two goods and three activities, the analogues to (2.10) to (2.11) would each consist of three variables and equations, and again indeterminacy would disappear.

The Sraffian dismissal of cases where $N + 1$ activities are in use rarely receives explicit defence. The rationale presumably is that the labour market is not required to clear, in which case there is no reason to suppose that w should equal one of the unusual values where $N + 1$ activities all earn the same rate of return.

The indeterminacy-reducing effect of factor-market clearing has already appeared. In the $N = 1$ example in Exhibit A we saw the dimension of indeterminacy drop from 2 to 1 when two activities are in use rather than one. Indeed, that example and the present example are essentially the same: r and the rental rate on land were the endogenous price variables in Exhibit A whereas r and the wage are the endogenous price variables here (w also appears in Exhibit A but we treated w as a parameter and ignored labour-market clearing). And just as in Exhibit A, the one-activity-in-use equilibrium requires a special configuration for (e_c, e_ℓ): when one activity i is in use and $(w, 1 + r) \gg 0$ we must have $e_c = e_\ell a_{1i}/\ell_i$. Similar conclusions hold when $N > 1$ (see Section 3).

Sraffians cannot have it both ways: if the case against an extra dimension of indeterminacy in the presence of land – that the special resource configurations are unlikely – is compelling, then consistency would seem to demand rejection of the single dimension of indeterminacy in the classical Sraffa setting. Of course, one may argue instead that labour unlike land is traded in a market that does not clear. But then the indeterminacy of the wage becomes an assumption rather than a conclusion: in any model where the labour market does not clear, the wage can be treated as a free parameter. We will expand on this point in the next section.

Gathering our exhibits together, we can summarize concisely what determines the extent of indeterminacy. Counting labour as an input, the dimension of indeterminacy equals the difference between the number of positively priced (hence fully utilized) inputs and the number of activities in use (see FACTOR PRICES IN GENERAL EQUILIBRIUM). In basic Sraffian indeterminacy, $N + 1$ inputs are used by N activities: hence 1 dimension of indeterminacy. In the $N = 1$ extensive-rent example with 1 activity in use, 3 inputs are used by 1 activity: 2 dimensions of indeterminacy. In the extensive-rent example where l types of land are in use and all l are fully utilized, $N + 1 + l$ inputs have a positive price and are used by $N + l - 1$ activities: again 2 dimensions of indeterminacy. In the joint production example with 2 goods, 3 inputs have a positive price: 2 dimensions of indeterminacy occur when 1 activity is in use and 1 dimension of indeterminacy occurs when 2 activities are in use. Finally, in our choice-of-activities example indeterminacy disappears when 2 positively priced inputs are used by 2 activities.

As we will now see, a general equilibrium account of factor-price indeterminacy also reports a dimension of indeterminacy equal to the difference between the number of scarce inputs and the numbers of activities in use.

3 Sraffian indeterminacy with explicit market clearing

The forces of supply and demand have been nipping at the heels of Sraffian indeterminacy: the labour market clearing requirement in Exhibit C sometimes eliminates indeterminacy, and our informal appeals to output demand functions appeared to eclipse the extra indeterminacy that arose in Exhibits A and B. So perhaps a careful inclusion of market clearing for all goods will snuff all the indeterminacy out. This turns out not to be the case.

We include labour among the markets that are required to clear. The only formal feature of labour in Sraffian models that distinguishes it from land is that homogenous labour is used in every sector whereas a specific type of land need not be. (In reality, different varieties of labour are used in different industries and some type of land is used in every industry.) So we treat labour (and stocks of other inputs) as we have previously modelled land: for a positive price to rule, demand must equal supply. Labour markets are distinctive of course – labour can require an efficiency wage to induce effort, wage contracts can serve as decades-long insurance contracts, workers can be in unions, and so forth – and perhaps these special traits lead labour markets not to clear. But if we simply exempt labour from market clearing by fiat, one purpose of the Sraffa model is undermined. Wage indeterminacy will obtain whenever the labour market does not have to clear – whether or not capital aggregates, relative prices are constant through time, or linear activities describe technology. Models that allow the labour market not to clear in effect *assume* that markets do not pin down the distribution of income; they do not demonstrate that principle.

We now distinguish explicitly between material goods when they are inputs at an earlier point in time and outputs at a later point. Two periods will be enough; material inputs and labour will be supplied inelastically at time 1, and output sold at period 2. Relative prices will no longer be restricted to remain proportional through time. Relative prices that vary from period to period run counter to Sraffian tradition but are indispensable: if indeterminacy is to survive in the presence of market clearing, additional free price variables are imperative.

The prices of the N material inputs supplied at time 1 will be denoted $p^1 = (p_1^1, \ldots, p_N^1)$ while the prices of the goods sold at time 2 will be $p^2 = (p_1^2, \ldots, p_N^2)$. As in the basic Sraffa model, suppose just N activities are available, one for each produced good, and let y denote the activity levels. Output demand is given by the demand function

$$x(p^1, p^2, w, 1+r) = (x_1(p^1, p^2, w, 1+r), \ldots, x_N(p^1, p^2, w, 1+r)),$$

and the exogenous supply of labour and material inputs is given by e_ℓ and the N-vector e. Together these ingredients must obey Walras' law:

$$\frac{1}{1+r} p^2 \cdot x(p^1, p^2, w, 1+r) = \frac{1}{1+r} w e_\ell + p^1 \cdot e.$$

An equilibrium at which $(p^1, p^2, w, 1+r, y) \gg 0$ satisfies

$$p_i^2 = (1+r)(p_1^1 a_{1i} + \cdots + p_N^1 a_{Ni}) + w\ell_i, \quad i = 1, \ldots, N, \tag{3.1}$$

$$x_i(p^1, p^2, w, 1+r) = y_i, \qquad i = 1, \ldots, N, \tag{3.2}$$

$$a_{j1} y_1 + \cdots + a_{jN} y_N = e_j, \qquad j = 1, \ldots, N, \tag{3.3}$$

$$\ell_1 y_1 + \cdots + \ell_N y_N = e_\ell, \tag{3.4}$$

$$p_1^1 + \cdots + p_N^1 = 1, \tag{3.5}$$

$$p_1^2 + \cdots + p_N^2 = 1. \tag{3.6}$$

There are two normalizations, (3.5) and (3.6), since the model uses an interest rate r rather than present value prices. The explicit inclusion of demand ensures that the model takes into account the indeterminacy-reducing effect of demand that we saw in Exhibits A and B.

The equilibria described by the above equalities typically will exhibit indeterminacy. If we fix y at an equilibrium value, then the market-clearing equalities for the material inputs and labour, (3.3) and (3.4), will remain satisfied as $(p^1, p^2, w, 1+r)$ varies. Of the remaining $2N+2$ equalities, one of the remaining market-clearing or zero-profit equalities is redundant due to Walras' law. But since the remaining $2N+1$ equalities have the $2N+2$ endogenous variables $(p^1, p^2, w, 1+r)$, one dimension of indeterminacy will typically obtain. The qualification 'typically' is necessary because a rank condition must hold in order to prove indeterminacy via the implicit function theorem (Mandler, 1999).

Several points give the above reasoning a Sraffian flavour. First, just as in Sraffa's book, aggregate quantities remain fixed and hence the indeterminacy operates on prices alone (though as the prices consistent with the fixed aggregate quantities change, individual incomes and individual consumption vary). Of course, unlike Sraffa, we know that markets clear at the fixed aggregate quantities. Second, linear activities are essential. With a differentiable technology a given vector of aggregate quantities would be incompatible with multiple equilibrium price vectors (Mandler, 1997). Third, it is no accident that the present supply-and-demand model and the basic Sraffian model both display a single dimension of indeterminacy. The degree of indeterminacy is the same because first, with inelastic input supply the market-clearing conditions for inputs do not restrict prices, and second, the $N-1$ independent market-clearing equalities for output are exactly counterbalanced by the

$N - 1$ new second-period prices (we lose one price due to the added normalization (3.6)). This match between added equilibrium conditions and added prices variables means that if we open the door to the variations considered in the previous section – inelastically supplied land, joint production, choice of activities – the dimensions of indeterminacy of the two approaches will still coincide. In both models, the dimension will equal the difference between the number of inelastically supplied first-period factors that have a positive price and the number of activities in use.

Indeterminacy therefore need not always obtain when the Sraffa price equations are embedded in a supply-and-demand model. If more than one activity per produced good is available, then equilibria may have $N + 1$ rather than N activities in use. This possibility should come as no surprise since the $N = 1$ no-indeterminacy example in Exhibit C had two activities in use. Indeed if $N = 1$ and we introduced choice among activities, the present supply-and-demand model would be exactly the model in Exhibit C (the market-clearing equality omitted in Exhibit C is superfluous by Walras' law and (3.5)–(3.6) imply $p^1 = p^2$). Despite their neglect in the Sraffian literature, equilibria where the number of activities in use exceeds the number of produced goods are perfectly plausible if all factor markets are required to clear.

To complete the case for the compatibility of Sraffian indeterminacy and market clearing, we must deal with a famous counterargument that with overwhelming probability any equilibrium will have at least as many activities in use as positively priced factors (Mas-Colell, 1975; Kehoe, 1980) – we faced a similar argument in Exhibit A. Conditions (3.3) and (3.4) consist of $N + 1$ equalities in the N unknowns y; hence for almost every endowment (e, e_ℓ) there will exist no y that obeys these equalities. Consequently for these generic endowments there will be no equilibria satisfying (3.1)–(3.6). If only these N activities are available, then at the generic endowments one of the material inputs or labour will be in excess supply and have a 0 price. Hence for generic endowments the number of positively priced factors will not exceed the number of activities in use. But the seemingly unusual endowments (e, e_ℓ) at which (3.3) and (3.4) do have a solution can arise systematically (see FACTOR PRICES IN GENERAL EQUILIBRIUM and Mandler, 1995). The material endowments e are the outcome of past savings–investment decisions; agents will not knowingly accumulate so much of a material input that it ends up in excess supply. Even when resources can be used productively no matter how great their supply, the endowments where $N + 1$ resources are used by N activities can still arise – those endowments appear at kinks on the production possibilities frontier (for example, see Figure 1 where the material input is accumulated just to the point where the available land is fully utilized using only one activity). This view of capital as a set of accumulated goods rather than a random endowment of nature fits well with Sraffa's view of production as circular.

We have had to do some damage to the Sraffian tradition to embed its indeterminacy claims in a market-clearing model. Inelastic factor supply finds no echo in the Sraffa literature. Not all factors have to be supplied inelastically for indeterminacy to obtain – it would be enough if some subset of k factors with positive prices were supply inelastically and were used by fewer than k activities – but some must be.

More heretical from the Sraffian point of view, we have had to let relative prices vary across time periods. Time-varying prices allow output prices to clear the output markets without constraining input prices. The $N = 1$ case obscures this feature of the indeterminacy since then there are no relative output prices. But when $N > 1$ relative prices will be constant through time in a market-clearing model only if the economy is in a steady state, and steady states typically are determinate. For example, an overlapping generations model, even with a linear activity analysis technology, has locally unique steady states (Mandler, 1999). Indeed steady states will be determinate in virtually any model where markets clear and saving responds to the rate of return (including Marxian models where investment is increasing in the profit rate). There are two reasons. First, in any given period of a steady state, the prices of that period's given stocks of producible factors are constrained to equal the prices of the same factors being produced for the next period; the indeterminacy arguments we gave earlier therefore cannot be applied. Second, for factors that are not produced, endowment levels then *should* be seen as random parameters, and it would therefore be a fluke if some set of k such factors were used by fewer than k activities without one of them being in excess supply.

4 Sraffian instability

One may also read the Sraffian critique of neoclassical economics as arguing that the failure of capital to aggregate can lead the savings–investment market to be unstable. The case for instability relies on 'reverse capital deepening', where the ratio of the value of an economy's capital goods relative to the number of workers employed increases as a function of the interest rate. Consider a constant-relative-price Sraffa model with one or more activities available to produce each good, no joint production and hence no fixed capital, and where the economy is in a steady state. Set some consumption good to be the numéraire. Then, if we fix the economy's vector of outputs per worker, the ratio of the value of capital to labour is well-defined at any given r (except possibly at switch points). An increase in r can affect the value-of-capital to labour ratio in two ways. First, for any produced good j, a 'real effect' can change which activity produces j at lowest cost, which in turn will alter the vector of capital goods per worker used in the production of *j*. Second, even if the activities in use remain the same (and with the composition of output still fixed), a change in r will affect the relative prices of capital goods. This 'price effect' will typically change the value of the capital goods each worker uses. So although the real effect of an increased interest rate might lead to a decrease in the quantity of each capital good used per worker, the price effect can cause the value of capital per worker to rise; hence reverse capital deepening can occur. This possibility can appear in an economy with one consumption good and just one capital good. An increase in the interest rate will then lower the physical capital to labour ratio, but the price of the capital good can increase by enough that the ratio of the value of capital to labour rises (Bloise and Reichlin, 2009). Since this chain of events can happen with a single capital good, it is, strictly speaking, misleading to identify reverse capital deepening with the failure of capital goods to aggregate.

Here is the instability scenario. To ensure that Sraffian 'long-run' prices rule, we must assume that before and after a shock the economy is in a steady state, where relative prices are constant. If for simplicity there is no fixed capital, the steady state assumption implies that the value of capital per worker equals investment per worker. Consequently, with reverse capital deepening, investment per worker can exceed savings per worker when the interest rate is above its equilibrium value, thus pushing the interest rate higher, further away from equilibrium. Similarly investment per worker can fall short of savings per worker when the interest rate is below equilibrium, pushing the interest rate lower.

The difficulty with this reasoning is that there is no market where long-run or steady-state savings and investment meet. When a shock to savings or investment occurs, the only instability that could undermine the economy must appear in markets at some specific set of dates – presumably the markets concurrent with the shock. Those markets, however, can equilibrate only at the out-of-long-run-equilibrium prices that obtain when the economy with the pre-shock endowments of resources begins its transition to a new post-shock long run. Long-run prices are therefore of dubious pertinence to the stability issue.

Garegnani (2000) and Schefold (2005) have argued that Sraffian instability surfaces in market-clearing general equilibrium models as a failure of tâtonnements to converge to equilibrium prices. This innovation in the Sraffian agenda has cleared away the cobwebs from the well-rehearsed interchange where Sraffians grouse that capital goods do not aggregate and Walrasians reply that equilibria in the Arrow–Debreu model exist.

The traditional model of a tâtonnement does not apply to an economy with linear activities: whenever positive economic profits can be earned, firms will want to expand without bound, and hence excess demands and tâtonnement price adjustments will be ill-defined. The most detailed and specific proposal to embed Sraffian instability in a general equilibrium model, Schefold (2005), steps around this problem by assuming that output prices are always set so that no activity makes positive profits, and letting the tâtonnement operate only on factor prices, which are the primary object of interest. Given a vector of factor prices, one may calculate the prices for outputs that minimize costs and then the consumer demand for outputs that result from these factor and output prices. The profit-maximizing decisions of firms, assuming they produce these output levels, then determine factor demands, and the difference between factor demand and factor supply leads to a tâtonnement price adjustment. Even in this setting, factor demand can be multi-valued since there can be many cost-minimizing factor combinations that produce any given output vector. To tackle this problem, one may define the tâtonnement with a differential inclusion rather than a differential equation (Mandler, 2005, and for differential inclusions generally see Aubin and Cellina, 1984).

Goods are distinguished by the date at which they appear and are of two types, *factors* which are not produced and *outputs* which can be produced. Factors can include the initial period's stock of capital goods as well as various types and dates of labour, land, and raw materials. Technology is given by a matrix of linear activities A where each activity (a column of A) produces only one output but may use any of the non-produced factors and produced goods as inputs. We follow the standard sign convention where positive entries in A denote outputs and negative entries indicate inputs, index goods so that outputs come first and factors second, and assume that positive quantities of all outputs can be produced simultaneously. To permit intermediate goods, an output can have negative as well as positive entries in A. Let A_o denote the output rows of A and let A_f denote the factor rows of A. For an arbitrary vector of factor prices p_f, we may find the competitive output prices $p_o(p_f)$ by solving the cost minimization problem $\min_y -p_f A_f y$ subject to $A_o y \geq (1, \ldots, 1)$, $y \geq 0$, and setting $p_o(p_f)$ equal to the Lagrange multipliers at a solution to this problem. With output prices set in this way, consumers' output and factor excess demands become functions of p_f alone. Let $x_o(p_f)$ and $x_f(p_f)$ denote these functions, which we assume obey Walras' law. The demand for factors by firms is a x_f in the set

$$X_f(p_f) = \{x_f : x_f \text{ maximizes } (p_o(p_f), p_f) \cdot (x_o(p_f), x_f) \text{ subject to } (x_o(p_f), x_f) = Ay, y \geq 0\}.$$

As we mentioned, $X_f(p_f)$ may have multiple elements.

An equilibrium is a p_f such that $x_f(p_f) \in X_f(p_f)$. A *factor tâtonnement* is then a function $p_f(t)$, differentiable almost everywhere, such that when differentiable there is a $x_f \in X_f(p_f(t))$ with

$$\dot{p}_f(t) = x_f(p_f(t)) - x_f.$$

Due to the sign convention governing A and since $x_f(p_f)$ is an excess demand, $x_f(p_f)$ and x_f will both usually be negative.

Since the factors can appear at any date, the model can cover a classical Sraffa economy with just labour and produced goods as inputs, so long as the economy is finite-lived: the factors subject to the tâtonnement would consist of the initial period's stocks of capital goods and labour at all dates, while all capital goods that appear after the initial period would be classified as outputs.

While our initial story of reverse-capital-deepening instability was driven by responses of the value of capital to the interest rate, comparable stories are possible that refer just to the non-produced factors that appear in the above factor tâtonnement. Suppose in the classical Sraffa setting that some bundle of activities is cost-minimizing at both low and high r's and some other set of activities is cost-minimizing at intermediate r's (this is called 'reswitching'). And suppose further that a steady state with a small labour supply will use one of these sets of activities and a steady state with a large labour supply will use the other set. Then, if the economy initially is in a steady state at either a high or intermediate r and has the small labour supply, an exogenous shift to the large labour supply would lower r in a new steady state and hence raise w, hardly the intuitive price response to a supply increase. This tale compares steady states and tracks the movement of the wage through time, whereas all prices in a factor tâtonnement respond simultaneously to disequilibrium. Schefold (2005) nevertheless suggests that reswitching can lead a factor tâtonnement to be unstable.

Evaluation of this claim faces an immediate difficulty: no matter how well-behaved firms' factor demands are, consumer behaviour, which here appears as the factor supply function $x_f(p_f)$, can by itself lead to instability. To block this path, let us assume that demand obeys the weak axiom, the traditional tool used in general equilibrium theory to tame an exchange economy's demand function and ensure tâtonnement stability. In the present setting the weak axiom states that, for any p_f and p'_f, $p_o(p'_f)\cdot x_o(p_f)+p'_f\cdot x_f(p_f)\leq 0$ and $(x_o(p'_f),x_f(p'_f))\neq(x_o(p_f),x_f(p_f))$ imply $p_o(p_f)\cdot x_o(p'_f)+p_f\cdot x_f(p'_f)>0$. Just as in an exchange economy, the weak axiom implies that a factor tâtonnement is stable (Mandler, 2005). Thus, no matter how many potential capital theory paradoxes are packed into the technology, if price adjustments are guided by excess demand and demand obeys the weak axiom, stability obtains. In fact, if the weak axiom is satisfied then in a factor tâtonnement the distance between the out-of-equilibrium prices that the auctioneer calls out and any equilibrium price vector will decline monotonically.

A tâtonnement is a highly artificial model of how an economy responds to disequilibrium: price adjustments are governed by 'notional' consumer demands, which cannot be satisfied at nonequilibrium prices, rather than by rationed or constrained demands that could be. On top of this problem, an intertemporal tâtonnement requires the prices of goods that appear at different time periods to adjust simultaneously. Perhaps in a more realistic setting the paradoxes of capital theory will turn out to be a distinct source of instability – but the case remains to be made.

5 Back to growth theory

Sraffa saw the economy as embedded in time, with endowments of produced inputs determined by the accumulation of capital, and he modelled technology using the plausible primitive of linear activities, not production functions packaged with suspicious differentiability assumptions designed to make factor returns determinate. These points add up to an effective criticism of a supply-and-demand theory of factor pricing, but the details of the argument need to be rearranged. The impossibility of capital aggregation plays no role.

But does the Sraffian stress on capital aggregation at least serve as an effective criticism of growth theory and the parables of the Solow model? On the surface, the fact that in a comparison of steady states the value of capital per worker or consumption per worker can increase with the interest rate may appear to undermine boilerplate neoclassical maxims on how to allocate resources through time. For example, it might seem that increases in savings could raise interest rates and lower future consumption. Unfortunately, as in the analysis of stability, the Sraffian focus on steady-state comparisons and on the value of capital per worker misleads. The move from one steady state to another involves the adjustment of myriad individual consumption and savings decisions at multiple points in time: consequently the impact of a change in savings today on steady-state consumption can diverge from the impact on consumption at a specific future date with all other consumption levels held constant. And when capital goods and consumption goods are separate commodities, changes in the relative prices of capital goods can break the linkage between increases in savings – sacrifices of present consumption – and increases in the value of capital; thus an increase in savings that lowers r *and* the value of capital per worker is not remarkable.

Following Solow (1963), define an economy's rate of return between the present and some future date t as the return in consumption at t as present-day consumption is sacrificed. If there is one consumption good per period, the gross rate of return between the present and t is the ratio of the gain in consumption at t relative to the quantity of today's consumption forgone, holding consumption at all other dates fixed. No reference to the value of capital is involved. With this definition, the familiar neoclassical maxims reappear: since linear activities and free disposal imply a convex production possibilities set, a sacrifice of consumption today must lead to a weak increase in consumption at t (holding all other consumption levels fixed) and the rate of return between today and t must weakly diminish in the quantity of consumption sacrificed. Comparisons of steady states can also be cleansed of references to the value of capital, but here a less than impressive set of claims is available. If again there is one consumption good per period and we avoid settings with infinitely many agents, such as the overlapping generations model, then any increase in steady-state consumption per worker entails an increase of the amount of at least one of the capital goods used per worker, and the move to a steady state with higher consumption per worker requires a sacrifice of consumption at some set of dates prior to arrival at the new steady state. The no-free-lunch moral of neoclassical growth theory rears its head.

But beneath these broad conclusions lies a vein of caveats, so far unmined by the Sraffian movement. When technology is described by linear activities, the rate of return can differ depending on whether it is defined by decreases or increases in present consumption. The reason is that the production possibilities set may well exhibit a kink at precisely the consumption levels chosen either by private agents in a market economy or by a benevolent planner who maximizes a sum of utilities (see Figure 1, but read the axes as present consumption and date t consumption). In the market-economy case, an exact match between the market interest rate r and the rate of return can then fail to obtain, though r must lie between the lower bound given by the rate of return for small increases in present consumption and the upper bound given by the rate of return for small decreases. In the planning case, the mismatch is between agents' intertemporal rates of substitution and the technological rate of return. Neoclassical growth theory has largely ignored these discrepancies, perhaps because of the blinders of its long reliance on differentiable production functions. But as in static factor pricing, linear activities in a growth setting open up a conceptual gap between prices and material rates of return: r no longer has to align with the physical return to sacrifices in consumption.

The presence of a kink in a production possibilities set will hinge on the number of activities in use, just as with factor-price indeterminacy. An irony crops up here: it is only in an optimal growth exercise that maximizes agents' utilities that an allocation at which the production possibilities set is kinked normally would be selected. Consider a planner with access to linear activities and stocks of resources at dates from the present (period 1) to the distance future (period T) that can be used to produce a sequence of consumption levels (again one consumption good per period). Resources are inelastically supplied and an arbitrary number of intermediate capital goods is permitted. Any consumption sequence $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_T)$ that is on the economy's PPF must satisfy the property that, for any $t = 1, \ldots, T$, $\bar{x}_t$ solves the problem of maximizing x_t subject to $(x_t, (\bar{x}_i)_{i \neq t})$ being in the production possibilities set. Pick one such problem with $t > 1$ where, if we view $\bar{x}_1$ as a parameter, the solution $x_t(\bar{x}_1)$ is non-constant. Consider those activities in use at some initial solution whose usage levels change as $\bar{x}_1$ changes. If no subset of k of these activities utilizes or produces more than k goods at the initial optimum, then (barring flukes of the production coefficients) the function $x_t(\bar{x}_1)$ will be differentiable at the initial $\bar{x}_1$: the PPF is smooth. (The good x_t should not be included in the count of the number of goods utilized or produced.) In the remaining cases, the initial $\bar{x}_1$ is a point where $x_t(\bar{x}_1)$ is not differentiable and the PPF is kinked; here inputs are accumulated just to the point where some set of k activities uses or produces more than k goods. Since almost every $\bar{x}$ on the PPF does not sit at a kink, a planner who selects a consumption stream arbitrarily could safely ignore the nondifferentiable points and declare that given the selection $\bar{x}$ the forces of technology alone determine the marginal rate of intertemporal transformation between any two time periods. If furthermore this planner decentralized the economy's investment decisions to private entrepreneurs the planner would have to choose this rate as the market rate of interest. Curiously, the Sraffian hostility to utility maximization and substitution in consumption (see Exhibits A and B) also leads to the conclusion that a $\bar{x}$ at a PPF kink is an unusual event; hence the Sraffian view comes to the aid of the neoclassical identification of interest rates with rates of technological transformation. On the other hand a planner who maximizes a sum of agent utilities could well choose a consumption stream at a kink (see Exhibit A and Figure 1). While the Sraffian emphasis on linear activities serves as a welcome corrective to the neoclassical habit of assuming that any function or surface is differentiable, in the end it is utility functions that prevent a linear activities growth model from providing a purely technological determination of the interest rate.

As we have seen, this lesson goes beyond growth theory. Market economies also gravitate to kinks on PPFs. Consequently, even when an economy has a single consumption good per period, which allows consumption output to be modelled with an aggregate production function, that production function may well not be differentiable when evaluated at the factor endowments that arise in equilibrium. Empirical work that relies on a differentiability assumption – for example, the classical growth-accounting estimates of total factor productivity (Solow, 1957; Kendrick, 1961; Denison, 1962) – is therefore subject to coherent Sraffian criticism.

6 Conclusion

The Sraffian insistence on linear activities casts critical light on the instinctive neoclassical habit of assuming that interest rates and marginal rates of transformation must be equal and that production functions must be differentiable. Another more abstract Sraffian principle proves just as illuminating: economic activity is ongoing, not a one-time exchange among agents with disparate endowments and preferences. As we have seen, it is when the endowments of capital goods are determined by rational accumulation rather than by chance that factor-price indeterminacy can appear.

The Sraffian view of equilibrium, which revives earlier classical ideas, fits well with subsequent mainstream developments. Modern macroeconomics, both new Keynesian and new classical, has rejected models where the government's actions, such as an aggregate demand stimulus, systematically surprise agents; instead government actions are governed by a distribution that agents know. The new understanding of expectations is not driven by a belief that agents are never surprised or never hold an incorrect model of the economy but in order to pinpoint results that are immune to invalidation as agents learn and adapt to their environment, a precept close to the Sraffian view of equilibria as ongoing. The Sraffian perspective has wide application. For example, while the production of capital endowments by past equilibrium activity can lead to factor-price indeterminacy, a similar dependence of the present on past equilibrium decisions can eliminate some disturbing features of other brands of indeterminacy (Mandler, 2002). In overlapping-generations indeterminacy, market clearing is compatible with agents at the beginning of economic time unanimously anticipating any future price path that lies in a multidimensional set. But if one sees an equilibrium as ongoing, rather than commencing anew each period, the indeterminacy problem disappears after an equilibrium gets under way. The agents in an economy that has already followed an anticipated price path for a number of periods and that experiences no shock will continue to hold their previously formed expectations, and given those expectations equilibrium prices at each period will be locally unique.

MICHAEL MANDLER

See also **capital theory (paradoxes); determinacy and indeterminacy of equilibria; factor prices in general equilibrium; general equilibrium; neo-Ricardian economics; Sraffian economics.**

Note on the dimension of indeterminacy of the basic Sraffa model

We may rewrite (2.4) as $p(I-(1+r)A)=w\ell$, where A is the matrix whose ith column is a_i and $\ell=(\ell_1,\ldots,\ell_N)$. Due to the homogeneity of (2.4) in (p, w), we can replace (2.3) with w=1 without changing the relative prices $\frac{1}{\|p\|}p$ in any solution $(p,w,1+r)$ to (2.3) or (2.4) or changing the dimension of the set of solutions. If at a solution $(\bar{p},\bar{w},1+\bar{r})\gg 0$ to $p(I-(1+r)A)=\ell$, $I-(1+\bar{r})A$ has rank N, then, for r near $\bar{r}$, $p=\ell(I-(1+r)A)^{-1}$ solves $p(I-(1+r)A)=\ell$. Hence locally there is a one-dimensional set of solutions. If $I-(1+\bar{r})A$ has rank $<N$, then $\{p:p(I-(1+\bar{r})A)=\ell\}$ has dimension ≥ 1, and so locally the solution set contains a set of dimension ≥ 1.

Bibliography

Aubin, J. and Cellina, A. 1984. *Differential Inclusions*. Berlin: Springer-Verlag.

Bharadwaj, K. and Schefold, B. (eds.) 1990. *Essays on Piero Sraffa*. London: Unwin Hyman.

Bidard, C. 1997. Pure joint production. *Cambridge Journal of Economics* 21, 685–701.

Bidard, C. 2004. *Prices, Reproduction, and Scarcity*. Cambridge: Cambridge University Press.

Bloise, G. and Reichlin, P. 2009. An obtrusive remark on capital and comparative statics. *Metroeconomica*.

Denison, E. 1962. *The Sources of Economic Growth in the United States*. Washington, DC: Committee for Economic Development.

Eatwell, J. 1982. Competition. In *Essays in Classical and Marxian Political Economy*, eds. I. Bradley and M. Howard. London: Macmillan.

Garegnani, P. 1976. On a change in the notion of equilibrium in recent work on value and distribution. In *Essays on Modern Capital Theory*, eds. M. Brown, K. Sato and P. Zarembka. Amsterdam: North-Holland.

Garegnani, P. 2000. Savings, investment, and capital in a system of general intertemporal equilibrium. In *Critical Essays on Piero Sraffa's Legacy in Economics*, ed. H. Kurz. Cambridge: Cambridge University Press.

Hahn, F. 1982. The neo-Ricardians. *Cambridge Journal of Economics* 6, 353–74.

Harcourt, G. 1974. The Cambridge controversies: the afterglow. In *Contemporary Issues in Economics*, eds. M. Parkin and A. Nobay. Manchester: Manchester University Press.

Kehoe, T. 1980. An index theorem for general equilibrium models with production. *Econometrica* 48, 1211–32.

Kendrick, J. 1961. *Productivity Trends in the United States*. Princeton: Princeton University Press.

Kurz, H. and Salvadori, N. 1995. *Theory of Production: A Long-period Analysis*. Cambridge: Cambridge University Press.

Lippi, M. 1979. *I prezzi di produzione*. Bologna: Il Mulino.

Mandler, M. 1995. Sequential indeterminacy in production economies. *Journal of Economic Theory* 66, 406–36.

Mandler, M. 1997. Sequential regularity in smooth production economies. *Journal of Mathematical Economics* 27, 487–504.

Mandler, M. 1999. Sraffian indeterminacy in general equilibrium. *Review of Economic Studies* 66, 693–711.

Mandler, M. 2002. Classical and neoclassical indeterminacy in one-shot versus ongoing equilibria. *Metroeconomica* 53, 203–22.

Mandler, M. 2005. Well-behaved production economies. *Metroeconomica* 56, 477–94.

Marglin, S. 1984. *Growth, Distribution, and Prices*. Cambridge, MA: Harvard University Press.
Mas-Colell, A. 1975. On the continuity of equilibrium prices in constant-returns production economies. *Journal of Mathematical Economics* 2, 21–33.
Quadrio-Curzio, A. 1980. Rent, income distribution, and orders of efficiency and rentability. In *Essays on the Theory of Joint Production*, ed. L. Pasinetti. London: Macmillan.
Salvadori, N. 1982. Existence of cost-minimizing systems within the Sraffa framework. *Zeitschrift für Nationalökonomie* 42, 281–98.
Salvadori, N. 1986. Land and choice of techniques within the Sraffa framework. *Australian Economic Papers* 25, 94–105.
Salvadori, N. 1990. Comment. In *Essays on Piero Sraffa*, eds. K. Bharadwaj and B. Schefold. London: Unwin Hyman.
Salvadori, N. and Steedman, I. 1988. Joint production analysis in a Sraffian framework. *Bulletin of Economic Research* 40, 165–95.
Schefold, B. 1978a. Multiple product techniques with properties of single product systems. *Zeitschrift für Nationalökonomie* 38, 29–53.
Schefold, B. 1978b. On counting equations. *Zeitschrift für Nationalökonomie* 38, 253–85.
Schefold, B. 1988. The dominant technique in joint production systems. *Cambridge Journal of Economics* 12, 97–123.
Schefold, B. 1990. On changes in the composition in output. In *Essays on Piero Sraffa*, eds. K. Bharadwaj and B. Schefold. London: Unwin Hyman.
Schefold, B. 1997. *Normal Prices, Technical Change, and Accumulation*. London: Macmillan.
Schefold, B. 2005. Reswitching as a cause of instability of intertemporal equilibrium. *Metroeconomica* 56, 438–76.
Solow, R. 1957. Technical changes and the aggregate production function. *Review of Economics and Statistics* 39, 312–20.
Solow, R. 1963. *Capital Theory and the Rate of Return*. Amsterdam: North-Holland.
Sraffa, P. 1960. *Production of Commodities by Means of Commodities*. Cambridge: Cambridge University Press.
Steedman, I. 1976. Positive profits with negative surplus value: a reply to Wolfstetter. *Economic Journal* 86, 873–6.

s-S models

The s-S model is the canonical model of inaction arising from costs of adjustment. Individuals do not always react to changes in their environment. Consumers rarely buy a new house or car after every fluctuation in their permanent income. Firms often leave prices fixed for months even though information is arriving at a much greater frequency. Fixed costs of adjustment provide a natural explanation of this inertial behaviour. If an agent faces a fixed cost to taking some action and if the loss to non-adjustment is small in the neighbourhood of the optimal choice, then it will pay to leave things be until the benefits of adjustment exceed the costs. Bar-Ilan and Blinder (1996) call this the 'optimality of usually doing nothing'.

The term s-S derives from inventory theory. In Arrow, Harris and Marschak's (1951) seminal paper, a firm allows its inventory holdings to decline below a level s before placing an order that replenishes inventories to a level S. Subsequently, the term s-S has come to denote an entire class of models of discrete and infrequent adjustment in which the optimal strategy is characterized by a set of triggers and targets. s-S models have been applied to explain inertia in a variety of microeconomic settings, including money demand, cash management, pricing, durable goods, and investment. In macroeconomics, the principal application has been to provide microfoundations for price stickiness and thereby the real effects of money. This is the menu cost model of price stickiness.

Microeconomics: the basic idea

The hallmark of the s-S policy is the combination of inaction and discrete adjustment. The basic idea can be simply illustrated in a static setting. Consider an agent who must choose x to minimize some twice differentiable, concave payoff function $\pi(x)$. The agent is endowed with a value x_0. The wrinkle is that there is a fixed cost k to changing x from its initial value. The optimal policy which balances the costs and benefits of adjustment is illustrated in Figure 1. If x_0 is less than S_L or greater than S_H, the benefit of adjusting to S^* outweighs the fixed cost of adjustment k. If x_0 is between S_L and S_H , inaction is optimal, and consequently $[S_L, S_H]$ is referred to as the range of inaction. The points S_L and S_H are, respectively, the upper and lower adjustment triggers. S^* is the adjustment target.

The combination of inaction and discrete adjustment is a direct result of the fixed cost of adjustment k. If instead the cost of adjustment were some twice differentiable convex function $c(x - x_0)$ with $c(0) = c'(0) = 0$, then there would be no inaction. Since the marginal cost of adjustment is zero at x_0, it would always be optimal to move closer to S^*.

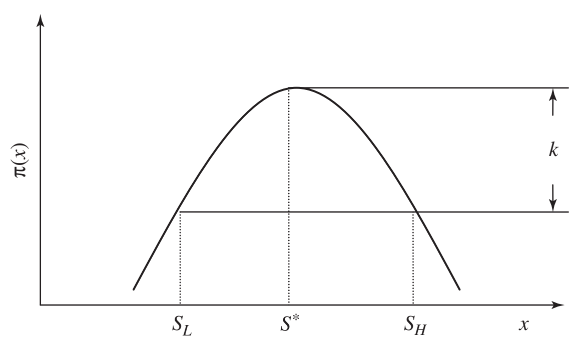

Figure 1 An s-S policy